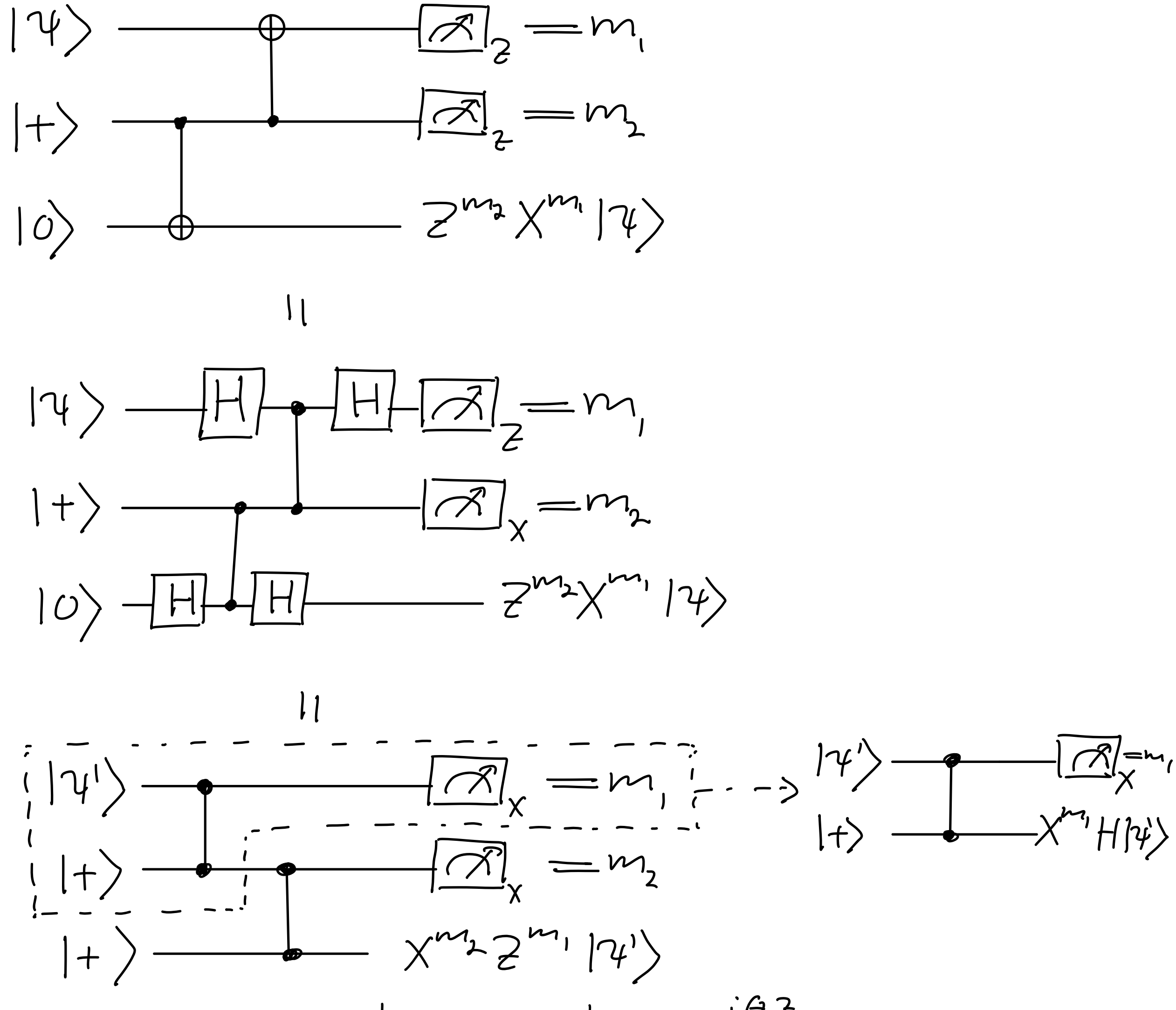
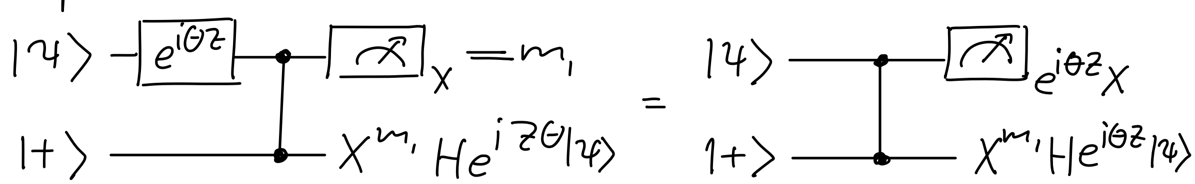
We can decompose the teleportation circuit into two elementary teleportations:



A single-qubit Z rotation $e^{i\theta Z}$ can be implemented in a teleportation-based way:



2) Now we are ready to formulate MBQC

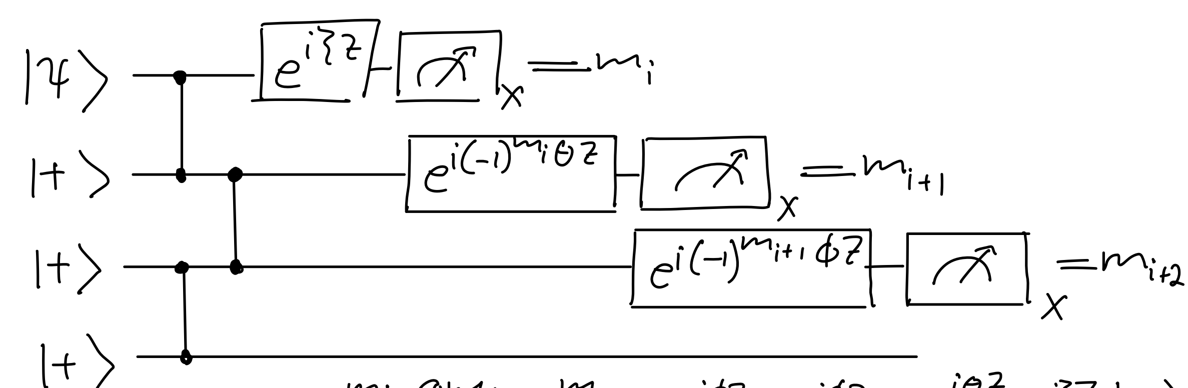
Decompose single-qubit unitary U (up to a phase) as:

$$U = He^{i\phi Z} e^{i\theta X} e^{i\xi Z}$$
$$= He^{i\phi Z} He^{i\theta Z} He^{i\xi Z}$$

$\longrightarrow$ arbitrary single-qubit unitary operation can be performed by a sequence of one-bit teleportations!

We have to take care of the byproduct Pauli operators depending on the measurement outcomes:

$$U = X^{m_{i+2}} He^{i\phi' Z} X^{m_{i+1}} He^{i\theta' Z} X^{m_i} He^{i\xi Z}$$
$$= X^{m_{i+2} \oplus m_i} Z^{m_{i+1}} He^{i(-1)^{m_{i+1}} \phi' Z} He^{i(-1)^{m_i} \theta' Z} He^{i\xi Z}$$



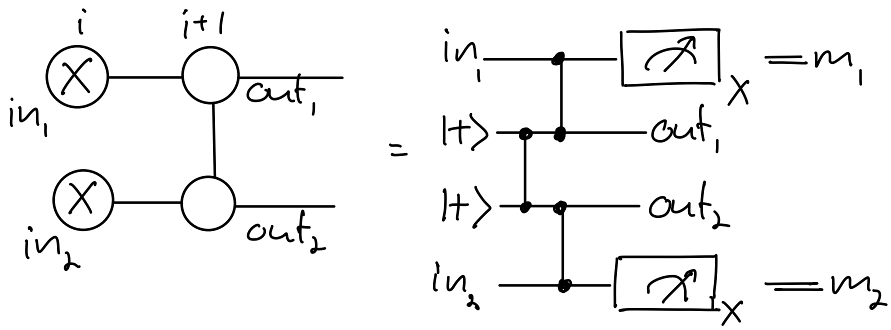
$$X^{m_{i+2} \oplus m_i} Z^{m_{i+1}} He^{i\phi Z} He^{i\phi Z} He^{i\theta Z} He^{i\xi Z} |\psi\rangle$$

By choosing $\theta' = (-1)^{m_i}\theta$ and $\phi' = (-1)^{m_{i+1}}\phi$, depending on previous measurement outcomes, U can be forwarded

$\longrightarrow$ "feed forward"

3) Measurement-based two-qubit gate:

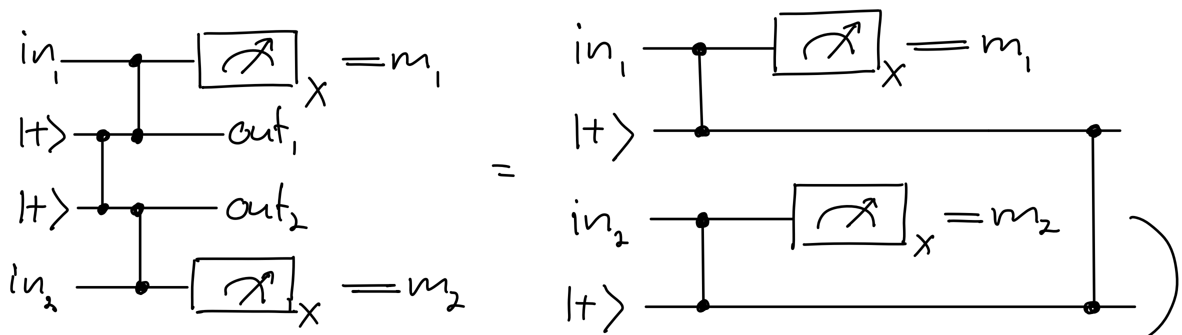
resource state is cluster state:



(recall graph state

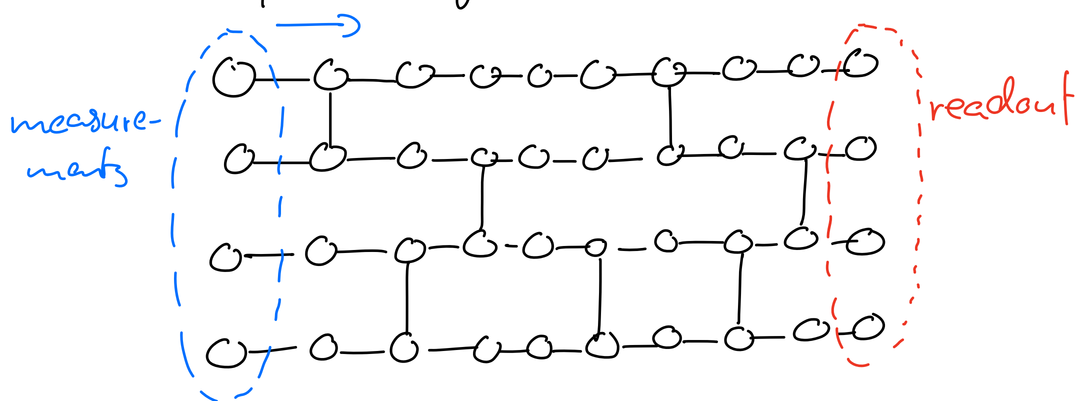
$$|G\rangle = \prod_{(i,j)\in E} \Lambda(Z)_{i,j}\, |+\rangle^{\otimes |V|} \;)$$

Now observe that



$$(X_2 Z_1)^{m_2}(X_1 Z_2)^{m_1}\,\Lambda_{1,2}(Z)\,H_1 H_2 |\psi\rangle_{1,2}$$

More generally, one can prepare an n-qubit graph-state :



→ perform measurments starting from the left-most column and successively to the right

→ simulate universal quantum computation by measurements !

§ 2.7 Quantum Error Correction Codes

1) Three-Qubit Bit-Flip Code:

Consider stabilizer generators

$$S_1 = Z_1 Z_2, \quad S_2 = Z_2 Z_3$$

$\rightarrow$ stabilizer subspace is spanned by

$$|0_L\rangle = |000\rangle, \quad |1_L\rangle = |111\rangle$$

define

- logical Pauli-X operator $L_X \equiv X_1 X_2 X_3$
- logical Pauli-Z operator $L_Z \equiv Z_1$

Consider a bit flip error with an error probability p:

$$\mathcal{E}_i \rho = (1-p)\rho + p X_i \rho X_i$$

acting successively on initial state

$$|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$$

$$\rightarrow \mathcal{E}_1 \circ \mathcal{E}_2 \circ \mathcal{E}_3 |\psi_L\rangle\langle\psi_L|$$
$$= (1-p)^3 |\psi_L\rangle\langle\psi_L| + p(1-p)^2 \sum_i X_i |\psi_L\rangle\langle\psi_L| X_i$$
$$+ \mathcal{O}(p^2)$$

error X_i maps code to an orthogonal subspace

→ perform projective measurements onto orthogonal subspaces,

$P_K^{\pm} = (I \pm S_K)/2$

→ measurement result is called "error syndrome"

4 error syndromes :

$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111|$ no error

$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011|$ bit flip on qubit one

$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101|$ bit flip on qubit two

$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110|$ bit flip on qubit three

suppose corrupted state is $a|100\rangle + b|011\rangle$

→ $\langle \psi | P_1 | \psi \rangle = 1$

<u>Note</u>: the syndrome measurement preserves the stat $\alpha|0_L\rangle + \beta|1_L\rangle$

Recovery:

$$\mathcal{R} \circ \mathcal{E}_1 \circ \mathcal{E}_2 \circ \mathcal{E}_3 |\psi_L\rangle\langle\psi_L|$$
$$= \left[(1-p)^3 + 3p(1-p)^2\right] |\psi_L\rangle\langle\psi_L| + \mathcal{O}(p^2)$$

where the recovery operator is given by

$$\mathcal{R}\rho = P_1^+ P_2^+ \rho P_2^+ P_1^+ + X_1 P_1^- P_2^+ \rho P_2^+ P_1^- X_1$$
$$+ X_2 P_1^- P_2^- \rho P_2^- P_1^- X_2 + X_3 P_1^+ P_2^- \rho P_2^- P_1^+ X_3$$

"fidelity" between a pure and a mixed state is given by $\sqrt{\langle\psi|\rho|\psi\rangle}$

without error correction:

$$\rho = (1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X$$

$$\Rightarrow \quad F = \sqrt{\langle\psi|\rho|\psi\rangle} = \sqrt{(1-p) + p\underbrace{\langle\psi|X|\psi\rangle\langle\psi|X|\psi\rangle}_{\geq 0}}$$

and $= 0$ for $|\psi\rangle = 0$

$$\Rightarrow \quad F \geq \sqrt{1-p}$$

with error correction we get

$$F = \sqrt{\langle\psi|\rho|\psi\rangle} \geq \sqrt{(1-p)^3 + 3p(1-p)^2}$$

$\rightarrow$ improved for $p < \frac{1}{2}$!